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The analytical treatment of storm surges

by

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1. Quoting the words of the late van Dantzig in an address for the international congress of mathematicians held in 1954 at Amsterdam, "the hydrodynamical problems posed to us in connection with flood prevention offer a typical instance where the job of a mathematician does not consist of just "feeding a machine", and which, although it ultimately certainly will require large scale computing, has need of what might be called "large scale mathematics"".

The investigation of the effect of wind upon the level of a shallow sea raises a great number of mathematical problems. In this paper a few typical problems will be selected and in particular it will be shown what can be reached by analytical methods. These methods may be indicated by the following headings: Boundary value problems in partial differential equations; eigenvalue problems, theory of complex functions and conform mapping; singular integral equations of the Cauchy type and of the Wiener-Hopf type; Laplace and Fourier transforms.

The above-mentioned analytical tools are most effective when the mathematical model under consideration is of a relatively simple kind. The analytical approach is aimed at understanding the influence of the parameters of the model and of the singularities of the solution. The analytical approach is aimed at deriving general laws and not so much at concrete numerical results.

The present-day powerful numerical tools may deal with far more complex situations. However, here it is difficult to assess the influence of the parameterssince this would require numerous repetitions of the same calculation with each time a different combination of values. With three or more degrees of freedom this would lead to an almost impossible task.

Thus the analytical approach and the numerical approach yield information of different natures. It will be clear that these two approaches may profit very much from each other. The laws discovered by analytical means may be numerically checked and, guided by the understanding gained by the analytical approach, the numerical calculation may be carried out in a much more economical way.

We consider here a shallow sea S with a uniform depth h bounded by a coast C_1 and a very deep ocean C_2. The differential equations are as follows

$$(1.1)\qquad \begin{cases} \dfrac{\partial \vec{w}}{\partial t} + \lambda \vec{w} + \vec{\Omega} \times \vec{w} + gh\,\nabla \zeta = \vec{W}(x,y,t) \\ \nabla \cdot \vec{w} + \dfrac{\partial \zeta}{\partial t} = 0 \end{cases} ,$$

where $\vec{w}$ is the vector of the total stream, $\vec{W}$ the wind stress acting on the surface, ζ the elevation of the surface, λ a constant accounting for the bottom friction and $\vec{\Omega}$ a constant vector representing the Coriolis effect.

The boundary conditions are as follows:

$$(1.2)\qquad \vec{w} \cdot \vec{n} = 0 \qquad \text{at } C_1$$

$$(1.3)\qquad \zeta = 0 \qquad \text{at } C_2 \quad .$$

To this we may add some initial conditions stating that at t=0 stream and elevation are known.

We note that the model is a linear one so that the important superposition-principle holds according to which the solution for an arbitrary wind field may be synthesized from a Green's function, i.e. the solution for a point source disturbance $\vec{\delta}(x,y,t,x_o,y_o,t_o)$.

An explicit analytic solution can be attained only when further simplifications are introduced.
The nature of these simplifications is indicated in the following scheme

- region
  - rectangle
    - full plane
    - half plane
    - strip
  - wedge
  - circle
- time
  - stationary
  - periodic
    - free motion
    - external periodic force
  - aperiodic

In the next section we shall consider a few aspects of the analytical treatment of the stationary problem. The last section will deal with a particular case of the non-stationary problem.

2. For the stationary situation the equations (1.1) may be written in Cartesian coordinates as follows

$$(2.1)\qquad \begin{cases} \lambda u - \Omega v + \zeta_x = U(x,y) \\ \lambda v + \Omega u + \zeta_y = V(x,y) \\ u_x + v_y = 0 \end{cases}$$

A stream function $\phi(x,y)$ may be introduced by means of

$$(2.2)\qquad \lambda u = -\phi_y \quad , \quad \lambda v = \phi_x \quad .$$

The lines of constant ϕ are streamlines.
Then it follows from (2.1) that

$$(2.3)\qquad \Delta \phi = R \quad ,$$

where $R=V_x-U_y$ is the rotation of the wind field. The boundary conditions become

$$(2.4)\qquad \phi = 0 \quad \text{at } C_1$$

$$(2.5)\qquad \frac{\partial \phi}{\partial n} - \frac{\Omega}{\lambda}\frac{\partial \phi}{\partial s} = S \text{ at } C_2 \quad ,$$

where $S=\vec{W}.\vec{s}$ is the component of the wind field along the ocean boundary. We may state the following result

Theorem

If the rotation of the windfield vanishes identically and if there is no wind along the ocean boundary there is no stream in the stationary situation.

Proof

The problem (2.3),(2.4),(2.5) has only the trivial solution $\phi \equiv 0$.

Corollary

A stationary rotation-free windfield on a lake induces no stream.

We give the following simple example which is of importance for the behaviour of the North Sea. We consider the rectangular region $0 < x < a$. $0 < y < b$ where $x=0$, $x=a$, $y=0$ are coasts and $y=b$ is an ocean boundary. The stationary wind field

(2.6) $\quad U=0 \quad V=-1$

will induce no stream. In fact (2.1) has the solution

(2.7) $\quad u=v=0 \quad , \quad \zeta=b-y \quad .$

On the contrary the stationary wind field

(2.8) $\quad U=1 \quad V=0$

does create a stationary stream the determination of which is not quite elementary.

By virtue of the superposition principle the general problem (2.3),(2.4),(2.5) may be reduced to the somewhat simpler problem where the wind field is reduced to a stationary point source. The corresponding Green's function $G(x,y,x_o,y_o)$ is determined by

(2.9) $$\Delta G = -\delta(x-x_o)\,\delta(y-y_o)$$

with homogeneous conditions at C_1 and C_2.

Calling in the aid of the theory of complex analytic functions we put

(2.10) $$G=\mathrm{Re}\,L(z,z_o)\ ,\quad z=x+iy\ ,\quad z_o=x_o+iy_o\ ,$$

where $L(z,z_o)$ is an analytic function of z and, with the sole exception of a logarithmic pole at z_o, holomorphic in the sea-region. More accurately the behaviour of L near z_o is as follows

(2.11) $$L(z,z_o)=-\frac{1}{2\pi}\ \ln(z-z_o)+O(1).$$

The boundary conditions (2.4) and (2.5) may be translated as follows

(2.12) $$\mathrm{Re}\ L=0 \quad \text{at } C_1$$

(2.13) $$\mathrm{Im}\ e^{-i\gamma} L=0 \quad \text{at } C_2$$

with γ given by

(2.14) $$\gamma = \mathrm{arctg}\,\frac{\Omega}{\lambda}\ .$$

A few particular cases will be listed below.

a Full plane

(2.15) $$L= -\frac{1}{2\pi}\ln(z-z_o)\ ,\quad G= -\frac{1}{2\pi}\ln\sqrt{(x-x_o)^2+(y-y_o)^2}\ .$$

b Half-plane $y>0$ with coast at $y=0$

(2.16) $$L= -\frac{1}{2\pi}\ln\frac{z-z_o}{z-\bar{z}_o}\ ,\quad G= -\frac{1}{4\pi}\ln\frac{(x-x_o)^2+(y-y_o)^2}{(x-x_o)^2+(y+y_o)^2}\ .$$

c Strip $0<y<\pi$ with coasts at $y=0$ and $y=\pi$
conformal map $z \rightarrow e^z$

(2.17) $$L= -\frac{1}{2\pi}\ln\frac{e^z-e^{z_o}}{e^z-e^{\bar{z}_o}}\ ,\quad G= -\frac{1}{4\pi}\ln\frac{\mathrm{ch}(x-x_o)-\cos(y-y_o)}{\mathrm{ch}(x-x_o)-\cos(y+y_o)}$$

d Half-plane $y>0$ with coast at $y=0$, $x>0$ and ocean at $y=0$, $x<0$

(2.18) $$L=\frac{1}{2\pi}\int_z^{\infty} s^{-\frac{1}{2}+\frac{\gamma}{\pi}}\left(\frac{z_o^{\frac{1}{2}-\frac{\gamma}{\pi}}}{s-z_o}-\frac{\bar{z}_o^{\frac{1}{2}-\frac{\gamma}{\pi}}}{s-\bar{z}_o}\right)ds\ .$$

The behaviour of G at the origin in polar coordinates is as follows:

(2.19) $$G=c\ r^{\frac{1}{2}+\frac{\gamma}{\pi}}\sin\left(\tfrac{1}{2}+\tfrac{\gamma}{\pi}\right)\theta$$

e North Sea model, rectangle $0<x<a$, $0<y<b$ with coasts at $x=0$, $x=a$, $y=0$ and ocean at $y=b$.
The solution can be obtained from (2.18) by means of conformal mapping with the mapping function

(2.20) $$z \rightarrow \wp(z-\omega_1-\omega_2)-e_3\ ,$$

where $\wp(z)$ is the Weierstrassian elliptic function with periods 2a and 2ib.

The behaviour at the ocean corners is as follows:

$$(2.21)\quad \begin{cases} G \approx r^{1+\frac{2\gamma}{\pi}} & \text{at } (0,b) \\ G \approx r^{1-\frac{2\gamma}{\pi}} & \text{at } (a,b) \end{cases}$$

where r is the local radius vector.

We note that according to (2.2) the stream vector is discontinuous at (a,b). A closer examination shows that u and v are of the form

$$(2.22)\quad \begin{cases} u = c\, r^{-\frac{2\gamma}{\pi}} \sin \frac{2\gamma}{\pi}\theta + \ldots \\ v = c\, r^{-\frac{2\gamma}{\pi}} \cos \frac{2\gamma}{\pi}\theta + \ldots \end{cases}$$

where θ is measured from the coast line x=a ($\theta=0$) to the ocean y=b ($\theta=\frac{1}{2}\pi$).

The North Sea model may also be treated by a direct method using series expansions. We shall demonstrate this for the particular wind field (2.8) although the method is of far greater generality. Thus we consider the following model of a rectangle x=0, x=π, y=0, y=b for which

$$(2.23)\quad \begin{cases} \Delta \phi = 0 & \\ \phi = 0 & \text{at } x=0,\ x=\pi,\ y=0 \\ \phi_y + \mathrm{tg}\,\gamma\, \phi_x = -1 & \text{at } y=b. \end{cases}$$

The solution will be of the following form

$$(2.24)\quad \phi = \sum_{n=1}^{\infty} c_n \frac{\mathrm{sh}\, ny}{\mathrm{sh}\, nb} \frac{\sin nx}{n}$$

where the coefficients are determined by the ocean condition. Taking advantage of the fact that in this model of the North Sea the rectangle is about twice as long as it is wide this condition gives approximately

$$(2.25)\quad -\cos\gamma = \sum_{n=1}^{\infty} c_n \sin(nx+\gamma)\ , \quad 0 < x < \pi\ ,$$

which may be called the expansion of a constant into an oblique Fourier series. This has led us to the systematic study of such series which is not only of importance in itself but which has useful applications in more difficult questions in connection with this North Sea model. Here we mention only the fact that the coefficients C_n of (2.25) are determined by

(2.26) $$\operatorname{cotg} \gamma \left(\frac{1-s}{1+s}\right)^{\frac{2\gamma}{\pi}} = \sum_{n=0}^{\infty} C_n s^n .$$

3. The non-stationary situation is determined by the equations (1.1) which we repeat here in Cartesian coordinates

(3.1) $$\begin{cases} (\frac{\partial}{\partial t} + \lambda) u - \Omega v + \zeta_x = U \\ (\frac{\partial}{\partial t} + \lambda) v + \Omega u + \zeta_y = V \\ \frac{\partial}{\partial t} \zeta + u_x + v_y = 0 \end{cases} .$$

Starting from a situation where everything is at rest at t=0 we may apply Laplace transformation according to

(3.2) $$\bar{\zeta}(x,y,p) = \int_0^{\infty} e^{-pt} \zeta(x,y,t)dt .$$

Then the equations (3.1) are transformed into

(3.3) $$\begin{cases} (p+\lambda)\bar{u} - \Omega \bar{v} + \bar{\zeta}_x = \bar{U} \\ (p+\lambda)\bar{v} + \Omega \bar{u} + \bar{\zeta}_y = \bar{V} \\ p\bar{\zeta} + \bar{u}_x + \bar{v}_y = 0 \end{cases} .$$

Elimination of $\bar{u}$ and $\bar{v}$ gives the following Helmholtz equation

(3.4) $$\bar{\zeta}_{xx} + \bar{\zeta}_{yy} - \kappa^2 \bar{\zeta} = \bar{F} ,$$

where

(3.5) $$\kappa^2 = p(p+\lambda) + \Omega^2 \frac{p}{p+\lambda} ,$$

and

(3.6) $$\bar{F} = \operatorname{div} \vec{W} + \frac{\Omega}{p+\lambda} \operatorname{rot} \vec{W} = (\bar{U}_x + \bar{V}_y) + \frac{\Omega}{p+\lambda} (\bar{V}_x - \bar{U}_y).$$

The boundary conditions are

$$(3.7) \qquad \frac{\partial \bar{\zeta}}{\partial n} + \frac{\Omega}{p+\lambda}\frac{\partial \bar{\zeta}}{\partial s} = \bar{W}_n + \frac{\Omega}{p+\lambda}\bar{W}_s \qquad \text{at } C_1$$

and

$$(3.8) \qquad \bar{\zeta} = 0 \qquad \text{at } C_2 .$$

From $\bar{\zeta}(x,y,p)$ the original $\zeta(x,y,t)$ may be derived by the inverse Laplace transformation

$$(3.9) \qquad \zeta(x,y,t) = \frac{1}{2\pi i}\int_L e^{pt}\,\bar{\zeta}(x,y,p)dp \quad ,$$

where L is a certain vertical line in the complex p-plane. By this formula the determination of ζ is reduced to the discussion of the singularities of the analytic function $\bar{\zeta}$. Assuming the existence of a stationary situation lim $\zeta(x,y,t)$ for $t \to \infty$ the ultimate elevation is determined by the residue of the pole p=0 of $\bar{\zeta}$. By way of illustration we consider the North-Sea model with the "northern" wind

$$(3.10) \qquad U=0 \qquad V= -(1-e^{-\varepsilon t}) .$$

Since ζ will not depend on x the equation (3.4) reduces to

$$(3.11) \qquad \bar{\zeta}_{yy} - \kappa^2\bar{\zeta} = 0$$

with the boundary conditions

$$(3.12) \qquad \bar{\zeta}_y = -\frac{\varepsilon}{p(p+\varepsilon)} \qquad \text{at } y=0$$

and

$$(3.13) \qquad \bar{\zeta} = 0 \qquad \text{at } y=b .$$

The solution is

$$(3.14) \qquad \bar{\zeta}(x,y,p) = \frac{\varepsilon}{p(p+\varepsilon)}\frac{\text{sh }\kappa(b-y)}{\kappa\text{ ch }\kappa b}$$

We note that p=0 is a pole with the residue b-y so that in agreement with (3.7)

$$(3.15) \qquad \zeta(x,y,\infty) = b-y \quad .$$

In view of the superposition principle it is sufficient

to solve the problem (3.4) with the boundary conditions (3.7) and (3.8) for a point-source disturbance.
The corresponding function of Green can explicitly be determined only for simple regions such as a half-plane, circle, strip, or angle. For other regions a Green's function might be used for which only a part of the boundary conditions is satisfied but then the problem eventually reduces to a singular integral equation or to an equivalent problem such as non-orthogonal expansions.

In order to show something of the relevant mathematical technique we consider the simplest case of a point-source disturbance on an infinite shallow sea.
Then (3.4) reduces to

$$\bar{\zeta}_{xx} + \bar{\zeta}_{yy} - \kappa^2 \bar{\zeta} = - \delta(x-x_o)\, \delta(y-y_o) \quad . \tag{3.16}$$

There are no boundary conditions but of course $\bar{\zeta}$ must be bounded at infinity.
The equation (2.16) has the solution

$$\bar{\zeta} = \frac{1}{2\pi} K_o\left(\kappa \sqrt{(x-x_o)^2+(y-y_o)^2}\right) . \tag{3.17}$$

Writing

$$R = \sqrt{(x-x_o)^2+(y-y_o)^2} \quad , \tag{3.18}$$

Laplace inversion gives the following results

a) $\lambda = \Omega = 0$

$$\zeta = \frac{\Theta(t-R)}{\sqrt{t^2-R^2}} \quad . \tag{3.19}$$

b) $\lambda = 0,\ \Omega \neq 0$

$$\zeta = \frac{\cos \Omega \sqrt{t^2-R^2}}{\sqrt{t^2-R^2}} \quad \Theta(t-R). \tag{3.20}$$

c) $\lambda \neq 0,\ \Omega = 0$

$$\zeta = e^{-\frac{1}{2}\lambda t} \frac{\mathrm{ch}\, \frac{1}{2}\lambda \sqrt{t^2-R^2}}{\sqrt{t^2-R^2}} \quad \Theta(t-R) . \tag{3.21}$$

If both λ and Ω differ from zero an explicit expression may be obtained but the solutions (3.20),(3.21) already show the influence of the parameters λ and Ω upon the final elevation.

For a stationary point-source disturbance which starts at t=0 and which may be described by

(3.22) $$F = -\Theta(t)\,\delta(x-x_o)\,\delta(y-y_o)$$

the corresponding solution may be obtained from that of the momentary disturbance by integration with respect to the time. For the case a) from (3.19) we derive e.g.

(3.23) $$\zeta = \int_R^t \frac{d\tau}{\sqrt{\tau^2-R^2}} = \left\{\ln(t+\sqrt{t^2-R^2})-\ln R\right\}\Theta(t-R) \quad .$$

The presence of even a single boundary condition leads to mathematical complications. The simplest case might be that of a half-plane sea $y > 0$ with a coast at the X-axis y=0. Then the solution is

(3.24) $$\bar{\zeta} = \frac{1}{2\pi}K_o(\kappa R) + \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{(p+\lambda)\mathrm{ch}w + i\Omega\,\mathrm{sh}w}{(p+\lambda)\mathrm{ch}w - i\Omega\,\mathrm{sh}w} \cdot \exp -\kappa\left\{i(x-x_o)\mathrm{sh}w + (y-y_o)\mathrm{ch}w\right\} dw$$

The inverse Laplace transformation may be carried out as before and we find for the case $\lambda = \Omega = 0$

(3.25) $$\zeta = \frac{\Theta(t-R)}{\sqrt{t^2-R^2}} + \frac{\Theta(t-R')}{\sqrt{t^2-R'^2}} \quad ,$$

where

(3.26) $$R' = \sqrt{(x-x_o)^2 + (y+y_o)^2} \quad .$$

The physical interpretation of this result is obvious. If $\lambda \neq 0$ and $\Omega = 0$ an analogous result is obtained. If $\lambda = 0$ and $\Omega \neq 0$ the inverse transformation may still be carried out in an explicit way but the final expression is slightly complicated and so it will be omitted here.

In order to show some other aspects of the non-stationary problem we consider the case of a strip $0 < y < b$ with a coast line y=0 and an ocean boundary. In order to simplify the discussion it will be assumed that the wind field is of the form U=0, V= $-\varphi(t)$ and that the resulting motion does not depend on the x-coordinate. Then the problem (3.4), (3.7) and (3,8) reduces to

(3.27) $$\frac{d^2\bar{\zeta}}{dy^2} - \kappa^2\bar{\zeta} = 0,$$

with the boundary conditions

(3.28) $$\frac{d\bar{\zeta}}{dy} = -\bar{\varphi}(p) \qquad \text{for } y=0 \quad ,$$

(3.29) $$\bar{\zeta} = 0 \qquad \text{for } y=b \quad .$$

The solution is obviously

(3.30) $$\bar{\zeta} = \bar{\varphi}(p)\,\frac{\operatorname{sh}\kappa(b-y)}{\kappa \operatorname{ch}\kappa b}$$

The eigenvalues of the problem are determined by the poles of $\bar{\zeta}$ i.e. the zeros of ch κ b.

Taking the values $\lambda=0.12$, $\Omega=0.6$, $b=2\pi$ of the North Sea model the following results are obtained.

a) $\lambda=0.12$, $\Omega=0$ $\qquad p^2+\lambda p+(\frac{1}{4}+\frac{1}{2}n)^2=0$, n=0,1,2,...

$p = -0.06 \pm i\ 0.24$

$p = -0.06 \pm i\ 0.75$

$p = -0.06 \pm i\ 1.25$ etc.

b) $\lambda=0$, $\Omega=0.6$ $\qquad p^2+\left\{\Omega^2+(\frac{1}{4}+\frac{1}{2}n)^2\right\}=0$, n=0,1,2,...

$p = \pm i\ 0.65$

$p = \pm i\ 0.96$

$p = \pm i\ 1.39$ etc.

c) $\lambda=0.12$, $\Omega=0.6$

$$p^3+2\lambda p+\left\{\lambda^2+\Omega^2+(\tfrac{1}{4}+\tfrac{1}{2}n)^2\right\} p + \lambda(\tfrac{1}{4}+\tfrac{1}{2}n)^2=0, \quad n=0,1,2,...$$

$p = -0.0173$, $\quad -0.11 \pm i\ 0.65$

$p = -0.067$, $\quad -0.09 \pm i\ 0.95$

$p = -0.091$, $\quad -0.07 \pm i\ 1.38$

This simple example shows an important hitherto not observed effect: If either λ or Ω vanishes the free motions are damped or undamped oscillations. However, if both λ and Ω differ from zero there results another sequence of real negative eigenvalues which corresponds to a sequence of free motions with a pure damping. The dominant eigenvalue is p= -0.0173 which is nearest to the origin. The effect upon the non-stationary behaviour of the sea may be illustrated by means of the special case of a step-function wind field $\varphi(t)=\theta(t)$. For the elevation at the coast y=0 we obtain

$$\tilde{\zeta}(x,0,p)=\frac{1}{p}\ \frac{\mathrm{th}\ \kappa b}{\kappa} \quad . \tag{3.26}$$

Laplace inversion gives with the above numerical values

$$\zeta(x,0,t)= 2\pi - 4.35\ e^{-0.0173\ t} +\ldots \tag{3.27}$$

The analysis of the North-Sea model is much more complicated and will not be discussed here.
The results are very similar to those of the simple strip model. Again the eigenvalues appear in groups of three corresponding to a pure damping and damped oscillations. The lowest eigenvalue is here p_o= -0.074. The response to a step-function wind field is here

$$\zeta(x,0,t) = 2\pi - 0.77\ e^{-0.074\ t} +\ldots \quad . \tag{3.28}$$

For further details the reader is referred to the papers on the North Sea Problem.

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